**COMBINATORICA**
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA **14** (1) (1994) 63–69

# NUCLEI OF POINT SETS OF SIZE $q+1$ CONTAINED IN THE UNION OF TWO LINES IN $PG(2,q)$

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*Received November 14, 1990*
*Revised February 20, 1991*

We give a complete classification for pairs $(\mathcal{N}(\mathcal{B}),\mathcal{B})$ where $\mathcal{N}(\mathcal{B})$ is the set of all nuclei of a set $\mathcal{B}$ of $q+1$ not collinear points contained in the union of two lines in a desarguesian plane $PG(2,q)$ of order $q$. We also obtain some new results concerning blocking sets of Rédei type and certain point-sets of type $[0,1,m,n]$ in $PG(2,q)$.

## 1. Introduction

Let $\mathcal{B}$ be any set of $q+1$ points in a desarguesian plane $\Pi$ of order $q$. A *nucleus* of $\mathcal{B}$ is a point $x$ such that each line of $\Pi$ through $x$ meets $\mathcal{B}$ exactly once. It follows that $x$ is not on $\mathcal{B}$. We shall denote by $\mathcal{N}(\mathcal{B})$ the set of all nuclei of $\mathcal{B}$.

The main result on nuclei due to Blokhuis and Wilbrink [1] states that if $\mathcal{B}$ admits at least $q$ nuclei then $\mathcal{B}$ is a line of $\Pi$. This result has been recently refined by A. A. Bruen and F. Mazzocca [4], and it has been also observed that some previous results on Galois geometries can be interpreted as results about nuclei. For instance, the main lemma in [8] is equivalent to the following theorem on nuclei: If the vertices of a triangle are nuclei of $\mathcal{B}$, then the sides meet $\mathcal{B}$ in three collinear points.

A typical problem on nuclei is to classify the possible configurations of $\mathcal{N}(\mathcal{B})$ for point sets $\mathcal{B}$ having assigned properties. In [6] all pairs $(\mathcal{N}(\mathcal{B}),\mathcal{B})$ have been determined under the assumption that $\mathcal{N}(\mathcal{B})\cup\mathcal{B}$ contains all points of an irreducible conic.

In this paper we deal with a set $\mathcal{B}$ of $q+1$ points not on a line but contained in the union of two lines. This special case is close related with blocking-sets of Rédei type as well as certain point sets of type $[0,1,m,n]$ in $PG(2,q)$. In Section 2, we give a complete classification of all pairs $(\mathcal{N}(\mathcal{B}),\mathcal{B})$ consisting of such a special set $\mathcal{B}$ and its *nuclei set* $\mathcal{N}(\mathcal{B})$. Our methods invole some properties of projectivity groups over Galois fields and hence do not work in an arbitrary plane.

AMS subject classification code (1980): 51 E 15

## 2. Preliminaries and some examples

Let $\Pi$ be a projective plane containing a proper subplane $\Pi_o$. Take any two lines $L_1$ and $L_2$ of $\Pi$ which belong to $\Pi_o$, and set $\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$ where $\mathcal{B}_1=L_1\cap\Pi_o$ and $\mathcal{B}_2=L_2-\Pi_o$. Clearly $\mathcal{B}$ is contained in the union of two lines, namely $L_1$ and $L_2$. It is easy to see that $\mathcal{N}(\mathcal{B})$ consists of all points of $\Pi_o$ not on the lines $L_1$ and $L_2$. This example can be generalized when $\Pi$ contains a chain of proper subplanes of $\Pi$, say $\Pi_o,\Pi_1,\dots,\Pi_s$, such that $\Pi_{i-1}\subset\Pi_i$ for $i=1,\dots,s$. In fact, if we take any two lines $L_1$ and $L_2$ of $\Pi$ belonging to $\Pi_o$ and set

(1) $L_i(0) = L_i\cap\Pi_o, \quad L_i(j) = (L_i\cap\Pi_j) - \Pi_{j-1}, \quad \text{for } i=1,2 \text{ and } j=1,\dots,s;$

If, for every decomposition $I=I_1\cup I_2$ of $I=\{0,1,\dots,s\}$ as the disjoint union of $I_1$ and $I_2$, we define

$$\mathcal{B}_i = \bigcup_{j\in I_i} L_i(j), \quad i=1,2. \tag{2}$$

then $\mathcal{B}=\mathcal{B}_1\cup\mathcal{B}_2$ is a set of $q+1$ points contained in $L_1\cup L_2$, and it turns out that $\mathcal{N}(\mathcal{B})=\Pi_o-(L_1\cup L_2)$.

From now on we assume that $\Pi=PG(2,q)$, $q=p^r$, $p$ prime and $r\geq 2$. We are going to show that the above examples are indeed special cases in a large collection of point sets $\mathcal{B}$ of size $q+1$ which are contained in the union of two lines and admit at least two nuclei.

Our methods involve some known properties of affinities and projectivities over a finite field $GF(q)$. We quote them in the text two paragraphs.

We shall denote by $\Sigma$ the group of all affinities of the affine line $AG(1,q)$ over $GF(q)$. Every subgroup of $\Sigma$ has order $tp^h$ where $t|p^d-1$ and $d|gcd[r,h]$; and conversely for such a pair $(t,h)$ there always exists a subgroup of order $tp^h$, see [5] Chapter XII. These subgroups can be described as follows: Let us consider $GF(q)$ as a vector space over a subfield $GF(q')$ of order $q'=p^d$, and take a subspace $\mathbf{B}$ of dimension $h_1$. Clearly $GF(q)$ contains $\mathbf{B}$ as an additive subgroup of order $p^h$ where $h=dh_1$, and for every $t\,|\,(p^d-1)$, $GF(q')$ has a multiplicative subgroup $\mathbf{A}$ of order $t$. Since $a\mathbf{B}=\mathbf{B}$ holds for every $a\in GF(q')$, the affinities $y'=ay+b$, where $a\in\mathbf{A}$, $b\in\mathbf{B}$, form a group $\mathbf{G}$ of order $tp^h$. For $a=1$ we have translation, and the translation group $\mathbf{T}$ of $\mathbf{G}$ is isomorphic to $\mathbf{B}$. Each other affinity in $\mathbf{G}$ is a dilatation, and the dilatations with a common center form a dilatation group $\mathbf{D}$ isomorphic to $\mathbf{A}$. Therefore, $\mathbf{G}$ is a frobenius group with nucleus $\mathbf{T}$ and complement $\mathbf{D}$, and hence $\mathbf{G}$ preserves $\mathbf{B}$ and acts on it as a transitive permutation group. Any other orbit $\mathbf{O}$ under $\mathbf{G}$ has length $tp^h$ in such a way that $\mathbf{G}$ acts on $\mathbf{O}$ as a regular permutation group. There exist $m=(p^{r-h}-1)/t$ orbits of the latter type, thus $\mathbf{B}\cup\mathbf{O}_1\cup\ldots\cup\mathbf{O}_m$ is the partition of $AG(1,q)$ into $\mathbf{G}$-orbits.

Take any two distinct lines $L_1$ and $L_2$ in $PG(2,q)$ and put $z=L_1\cap L_2$. For any point $u$ not on either line, we shall consider two perspectivities with center $u$: one $\pi_i$, sending $L_1$ to $L_2$ and another, say $p_u$, sending $L_2$ to $L_1$. Any product $\pi_u p_v$ maps the affine line $L_1-\{z\}$ onto itself. Precisely, $\pi_u p_v$ is an affinity and if $u$ is a left fixed and $v$ runs over all points, such products give all affinities, each exactly once. Similarly for the affine line $L_2-\{z\}$ and the products $p_u\pi_v$. Choose a frame for $L_1-\{z\}$ and a frame for $L_2-\{z\}$, and denote by $o$ the common point of the lines through their origins and their unit points, respectively. Then both perspectivities with center $o$ have the property that any two corresponding points posses the same coordinate. For a subgroup $\mathbf{G}$ of $\Sigma$, let $\mathbf{G}_i$ $(i=1,2)$ be the natural representation of $\mathbf{G}$ on the affine line $L_i-\{z\}$. Then $\mathbf{G}_2=\pi_o\mathbf{G}_1p_o$ and $\mathbf{G}_1=p_o\mathbf{G}_2\pi_o$. Furthermore, let $\mathbf{B}^i\cup\mathbf{O}^i\cup\ldots\cup\mathbf{O}^i_m$, $m=(p^{r-h}-1)/t$, be the partition into $\mathbf{G}_i$-orbits. Then $\mathbf{B}^1$ and $\mathbf{B}^2$ as well as $\mathbf{O}^1_i$ and $\mathbf{O}^2_i$ correspond by both $\pi_o$ and $p_o$.

Now pick either $\mathbf{B}^1$ or $\mathbf{B}^2$ and, for every $i$ $(1\le i\le m)$, one between $\mathbf{O}^1_i$ and $\mathbf{O}^2_i$. The union of these orbits together with $z$ is a set $\mathscr{B}$ satisfying our conditions, namely $\mathscr{B}$ consists of $q+1$ non collinear points each point of $\mathscr{B}$ lies on $L_1$ or $L_2$.

Clearly $\mathbf{G}_1$ preserves $\mathscr{B}_1=\mathscr{B}\cap L_1$. It may be, however, that the full affine group $\mathbf{H}_1$ mapping $\mathscr{B}_1$ onto itself is larger than $\mathbf{G}_1$. If this is the case, then a proper multiple $t'$ of $t$ divides $p^d-1$ and $\mathbf{H}_1$ consists of all affinities of the form $y'=ay+b$, $a\in\mathbf{A}'$, $b\in\mathbf{B}$, $a\neq 0$, where $\mathbf{A}'$ is the subgroup of order $t'$ is the multiplicative group of $GF(q)$. Furthermore $\mathbf{H}_1=\pi_o\mathbf{H}_1p_o$ is the full affine group preserving $\mathscr{B}_2=\mathscr{B}\cap L_2$.

Let $U$ be the set of all points $u$ such that $\mathbf{G}_1=\{p_u\pi_o\mid u\in U\}$, or equivalently, $\mathbf{G}_2=\{\pi_up_o\mid u\in U\}$. Note that $|U|=|\mathbf{G}|$. We are in a position to prove the following

**Proposition 1.** *Each point of $U$ is a nucleus of $\mathscr{B}$. But $U=\mathscr{N}(\mathscr{B})$ if and only if $\mathbf{G}_1$ is the full affine group preserving $\mathscr{B}_1$.*

**Proof.** Let $u\notin L_1\cup L_2$ be any point. For any line $L$ through $u$ which is different from $uz$, put $\ell_1=L\cap L_1$ and $\ell_2=L\cap L_2$. Clearly $\pi_u(\ell_1)=\ell_2$. Now assume $u\in U$. Then $\pi_u=\pi_og_1$ for a suitable element $g_1\in\mathbf{G}_1$. Hence, if $\ell_1\mathbf{B}^1$ then $\ell_2\in\mathbf{B}^2$, and also $\ell_1\in\mathbf{O}^1_i$ implies $\ell_2\in\mathbf{O}^2_i$. This means that $L$ meets $\mathscr{B}$ in exactly one point, hence we get $u\in\mathscr{N}(\mathscr{B})$. Conversely, let $u$ be a nucleus of $\mathscr{B}$. Then $p_u\pi_o$ is an affinity $g_1$ preserving $\mathscr{B}_1$. If $g_1\in\mathbf{G}_1$ then $u\in U$. Otherwise $\mathbf{G}_1$ is a proper subgroup of $\langle\mathbf{G}_1,g_1\rangle$ and so $g_1$ does not preserve all orbits $\mathbf{O}^1_i$. From this $u\notin U$ follows. Finally, if we make the assumption that an affinity $h_1\notin\mathbf{G}_1$ preserves $\mathscr{B}_1$, then $\mathscr{N}(\mathscr{B})$ turns out to be larger than $U$ since the center of the perspectivity $\pi_oh_1$ is a nucleus of $\mathscr{B}$ different from the points of $U$.

## 3. A classification for pairs $(\mathscr{B},(\mathscr{N}(\mathscr{B}))$

Let $\mathscr{B}$ be any set of $q+1$ non collinear points in $PG(2,q)$ contained in the union of two lines $L_1$ and $L_2$ such that $z=L_1\cap L_2$ belongs to $\mathscr{B}$. For $i=1,2$, put $\mathscr{B}_i=\mathscr{B}\cap L_i$ and assume that $|\mathscr{B}_i|\ge 2$. Let $\mathscr{N}(\mathscr{B})$ denote the set of all nuclei of $\mathscr{B}$. In general $\mathscr{N}(\mathscr{B})$ is empty. It is clear however that the point $P$ not on $L_1\cup L_2$ is a nucleus of $\mathscr{B}$ when the projection from $L_1$ on $L_2$ with center $P$ sends $\mathscr{B}_1-\{z\}$ to $L_2-\mathscr{B}_2$. Thus $\mathscr{N}(\mathscr{B})$ is not empty for a large family of sets $\mathscr{B}$.

Our aim is to show that $|\mathscr{N}(\mathscr{B})|\ge 2$ is possible only for very special choices of the point set $\mathscr{B}$ on $L_1\cup L_2$. Precisely, the only possibilities are the examples given in section 2.

Let $u$ be any point not on either line, and let $\pi_u$ (resp. $p_u$) denote the perspectivity with center $u$ which sends $L_1$ to $L_2$ (resp. $L_2$ to $L_1$). Put $\mathbf{G}_1=$

$\{\pi_u p_v \mid \forall u, v \in \mathcal{N}(\mathcal{B})\}$ and $\mathbf{G}_2 = \{p_v \pi_u \mid \forall u, v \in \mathcal{N}(\mathcal{B})\}$. Clearly $\mathbf{G}_i$ $(i=1,2)$ is a set of affinities of the affine line $L_i - \{z\}$ with the following property:

**Proposition 2.** *For $i=1,2$, $\mathbf{G}_1$ leaves $\mathcal{B}_i$ invariant.*

Actually we can say more:

**Proposition 3.** *For $i=1,2$, $\mathbf{G}_i$ is a group of affinities of the affine line $L_i - \{z\}$.*

**Proof.** Let $u$ be any fixed point $u \notin L_1 \cup L_2$. As is known, for any two points $w$ and $s$ not on either line, we can find a point $v$ such that $\pi_w p_s = \pi_u p_v$, and also a point $v'$ such that $\pi_w p_s = \pi_{v'} p_u$. To see that $\mathbf{G}_1$ is actually a group, it suffices to observe that if $u, w, s \in \mathcal{N}(\mathcal{B})$, then both $v$ and $v'$ are also in $\mathcal{N}(\mathcal{B})$. Similarly for $\mathbf{G}_2$.

From now on we fix a nucleus $o$ of $\mathcal{B}$. We can write

$$\mathbf{G}_1 = \{\pi_o p_v \mid v \in \mathcal{N}(\mathcal{B})\} \text{ and } \mathbf{G}_2 = \{p_v \pi_o \mid v \in \mathcal{N}(\mathcal{B})\}.$$

From this we infer

**Proposition 4.** *For any fixed nucleus $o$ of $\mathcal{B}$, we have both $\mathbf{G}_2 = p_o \mathbf{G}_1 \pi_o$ and $\mathbf{G}_1 = \pi_o \mathbf{G}_2 p_o$. Furthermore, $\mathbf{G}_1 \cong \mathbf{G}_2$.*

For $i=1,2$, choose a frame for $L_i - \{z\}$ in such a way that $\mathbf{G}_i$ coincides with a group $\mathbf{G}$ consisting of all affinities of equations $y' = ay + b$, $a \in \mathbf{A}$ and $b \in \mathbf{B}$, where $\mathbf{B}$ is an additive subgroup of order $p^h$ of $GF(q)$, $q = p^r$, and $\mathbf{A}$ is a multiplicative subgroup of order $t$ of $GF(q)$ such that $t \mid (p^{mcd[h,r]} - 1)$. Note that $\mathbf{G}_i$ is the full affine group mapping $\mathcal{B}_i$ $(i=1,2)$ onto itself because of Proposition 1. As in Section 2, put $m = (p^{r-h} - 1)/t$ and let $\mathbf{B}^i \cup \mathbf{O}_1^i \cup \ldots \cup \mathbf{O}_m^i$ be the partition of $L_i - \{z\}$ into $\mathbf{G}_i$-orbits.

By Proposition 4, the line joining a point of $L_1 - \{z\}$ with the point of $L_2 - \{z\}$ having the same parametric coordinate passes through $o$. Then $\mathbf{B}^1$ and $\mathbf{B}^2$ as well as $\mathbf{O}_i^1$ and $\mathbf{O}_i^2$ correspond by both $\pi_o$ and $p_o$. Since $o \in \mathcal{N}(\mathcal{B})$, this implies that $\mathcal{B}$ cannot contain both $\mathcal{B}^1$ and $\mathcal{B}^2$ as well as both $\mathbf{O}_i^1$ and $\mathbf{O}_i^2$ $(i-1, \ldots, m)$.

As $\mathcal{B}$ consists of $q+1$ points, it turns out that $\mathcal{B}$ contains either $\mathcal{B}^1$ or $\mathcal{B}^2$ and, for each $i$ $(1 \le i \le m)$, just one of $\mathbf{O}_i^1$ and $\mathbf{O}_i^2$. This proves that $\mathcal{B}$ is one of the examples given in section 2, and we get the following *canonical form*:

To any pair $(\mathcal{B}, \mathcal{N}(\mathcal{B}))$ there correspond an additive subgroup $\mathcal{B}$ of order $p^h$ and a multiplicative subgroup $\mathbf{A}$ of order $t$ in $GF(q)$ such that $a\mathbf{B} = \mathbf{B}$ holds for every $a \in \mathbf{A}$ (and hence $t \mid p^{gcd[r,h]} - 1$): Let $\mathbf{B} \cup \mathbf{O}_1 \cup \ldots \cup \mathbf{O}_m$, $m = (p^{r-h} - 1)/t$, be the partition of $GF(q)$ where $\mathbf{O}_i = \{ay_i + b \mid a \in \mathbf{A}, b \in \mathbf{B}\}$ and $y_1, \ldots, y_m$ are suitable element in $GF(q)$. Choose a frame in $PG(2,q)$ such that $L_1$ is the $x$-axis and $L_2$ is the line with equation $y = 1$ and put $\mathbf{B}^1 = \{(b,0,1) \mid b \in \mathbf{B}\}$, $\mathbf{B}^2 = \{(b,1,1) \mid (b \in \mathbf{B}\}$, $\mathbf{O}_i^1 = \{(c,0,1) \mid c \in \mathbf{O}_i\}$ and $\mathbf{O}_i^2 = \{(c,1,1) \mid c \in \mathbf{O}_i\}$. Now pick either $\mathbf{B}^1$ or $\mathbf{B}^2$ and, for every $i$ $(i \le i \le m)$, one between $\mathbf{O}_i^1$ and $\mathbf{O}_i^2$. The union of these point sets together with $L_1 \cap L_2$ is a set $\mathcal{B}$ of length $q+1$ contained in the union of $L_1$ and $L_{2'}$ and $\mathcal{N}(\mathcal{B})$ contains the set $\mathcal{U} = \{(-a, b, 1) \mid a \in \mathbf{A}, b \in \mathbf{B}\}$. The linear collineation group $\Gamma$ consisting of all maps $x' = x$, $y' = ay + b$, $a \in \mathbf{A}$, $b \in \mathbf{B}$ preserves $\mathcal{B}$, and $\mathcal{N}(\mathcal{B}) = \mathcal{U}$ if and only if $\Gamma$ is the full linear collineation group which preserves $\mathcal{B}$ and both $L_1$ and $L_2$.

We point out some special cases corresponding to special choices of **A** and **B**.

1. — If both **A** and **B** are trivial, then $\mathcal{N}(\mathcal{B})$ consists of a single point.

2. — If **A** is trivial, then $\mathcal{N}(\mathcal{B})$ is a set of $p^h$ points on the line $oz$.

3. — If **B** is trivial, then $\mathcal{N}(\mathcal{B})$ is a set of $t$ points on a line through $o$ but $z$. This is case, for instance, when $r=1$.

4. — Assume that **B** is the additive and **A** is the multiplicative subgroups of a subfield $GF(q_o)$ of $GF(q)$. Then $\mathcal{N}(\mathcal{B})$ together with $\mathbf{B}^1 \cup \mathbf{B}^2 \cup \{z\}$ forms a projective subplane $\Pi_o = PG(2,q_o)$. Furthermore, since **G** preserves any affine subline over a subfield of $GF(q)$ containing $GF(q_o)$, each set $L_i(j)$ given in (1) is the union of orbits under $\mathbf{G}_i$. Hence $\mathcal{B}=\mathcal{B}_1 \cup \mathcal{B}_2$ with $\mathcal{B}_1$ and $\mathcal{B}_2$ as in (2) turns out to be a special case of the present construction.

Finally we prove

**Proposition 5.** *Apart from the cases 1), 2), 3),* $\mathcal{N}(\mathcal{B})$ *is a set of type* $[0,1,t,p^h]$, $t=p^d-1$, $d\,|\,gcd[r,h]$, *with the following extra properties:*

(i) *all* $p^h$*-secants of* $\mathcal{N}(\mathcal{B})$ *are concurrent at the common point of* $L_1$ *and* $L_2$*;*

(ii) *each* $t$*-secant of* $\mathcal{N}(\mathcal{B})$ *meets both* $\mathbf{B}^1$ *and* $\mathbf{B}^2$.

**Proof.** For any point $u \in \mathcal{N}(\mathcal{B})$, the elements of $\mathbf{G}_1$ can be written as products $p_v \pi_u$ where $v$ runs over all points of $\mathcal{N}(\mathcal{B})$. Such a product is a translation on the affine line $L_1 - \{z\}$ is and only if $v$ is on line $uz$. From this we infer that $uz$ meets $\mathcal{N}(\mathcal{B})$ in precisely $p^h$ points. Otherwise we have a dilatation on $L_1 - \{z\}$ whose center is the common point $w_1$ of $L_1$ and $uv$. Here $w_1$ belongs to $\mathbf{B}^1$. Since any dilatation is contained in a maximal dilatation subgroup of order $t$, we see that $\mathcal{N}(\mathcal{B})$ and $uw$ have $t$ common points. Similarly for $\mathbf{G}_2$ and $\mathbf{B}^2$. Finally, if none of the cases 1), 2), 3) occurs, then $\mathcal{N}(\mathcal{B})$ admits both 1-secants as well as missing lines and hence it has type $[0,1,t,p^h]$.

## 4. Some applications of the classification theorem.

We begin with a problem related to blocking sets of Rédei type in $PG(2,q)$, i.e. blocking sets of size $q+k$ containing $k$ collinear points. The reason of this term comes from a closed connection between the existence problem for such blocking sets and one of the main results in Rédei's book ([7], p. 226), having to do with difference quotient in finite fields. This connection was pointed out by Bruen and we give a brief account here.

Given a set $S$ of $q$ non collinear points in $PG(2,q)$, if $L$ is a line missing $S$, then the *Rédei set* $R_S(L)$ consists of all points of $L$ lying on some secants of $S$. If $R_{\mathcal{S}S}(L)$ is properly contained in $L$, then $(S,L)$ is called a *Rédei pair*, and the set $S(L)=S \cup R_S(L)$ turns out to be a blocking set of Rédei type such that $k=|R_S(L)|$. Observe that if $(S,L)$ is a Rédei pair then $L$ contains a point $u$ which is a nucleus of $S \cup \{z\}$ where $z \in L$ is any point different from $u$. The converse is also true. In other words, there exists a line $L$ for which $(S,L)$ is a Rédei pair if and only if there exist two points $u$, $v$ not on $S$ such that $u$ is a nucleus of $S \cup \{v\}$.

Bruen raised following problem: *For which line* $L$ *missing* $S$ *is the blocking set* $S(L)$ *of minimal cardinality?* In [3], he proved that for any line $L$ such that

$|R_S(L)| < \frac{q}{2}+1$ holds, $S(L)$ is of minimal cardinality. Here we investigate the special case where $S$ is contained in the union of two lines.

**Proposition 6.** *Let $S$ be a set pf $q$ non-collinear points in $PG(2,q)$, $q=p^r$ and $p$ prime, contained in the union two lines $L_1$ and $L_2$.*

*If $z=L_1 \cap L_2$ is not on $S$ and $\mathcal{B}=S\cup\{z\}$, then*

(I) *a Rédei pair $(S,L)$ exists if and only if $\mathcal{B}$ has a nucleus;*
(II) *$(S,L)$ is a Rédei pair if and only if $L$ passes through $z$ and meets the set $\mathcal{N}(\mathcal{B})$ of all nuclei of $\mathcal{B}$.*
(III) *for any Rédei pair $(S,L)$, one has $|R_S(L)|=p^r+1-p^h$ where $h$ depends only on $S$.*

*If $z=L_1\cap L_2$ is on $S$, then*

(IV) *a Rédei pair $(S,L)$ exists if and only if there is a point $v\in(L_1\cap L_2)-S$ such that the set $\mathcal{B}=S\cup\{v\}$ has a nucleus;*
(V) *$(S,L)$ is a Rédei pair if and only if $L$ passes through $v$ and meets the set $\mathcal{B}(\mathcal{N})$ of all nuclei of $\mathcal{B}$.*
(VI) *for any Rédei pair $(S,L)$, one has $|R_S(L)|=p^r+1-t$ where $t$ is a divisor of $p^r-1$ depending only on $S$;*

**Proof.** Assume first that $z=L_1\cap L_2$ is not on $S$. Suppose that $(S,L)$ is a Rédei pair. If $u$ is any point of $L-R_S(L)$, then the lines joining $u$ to the points of $S$ are pairwise different. Since $|S|=q$, $uz$ turns out to be the unique line missing $S$. Therefore $L=uz$ and $u$ is a nucleus of $\mathcal{N}(\mathcal{B})$. Conversely, let $L$ be a line through $z$ which meets $\mathcal{N}(\mathcal{B})$. If $u\in L\cap\mathcal{B}(\mathcal{N})$, then $u\in L-R_S(L)$, thus $(S,L)$ is a Rédei pair. This completes the proof of both (I) and (II). We have also shown that if $(S,L)$ is a Rédei pair then $L-R_S(L)=L\cap\mathcal{N}(\mathcal{B})$. Therefore, (III) follows from (i) of Proposition (5). A similar argument proves the second part of our proposition.

Another application is concerned with a special family of point sets of type $[0,1,m,n]$. Let $S$ be a point set of size $(n-1)m+1$ in $PG(2,q)$ having all of the following three properties:

(x) $m>2$;
(xx) *all $n$-secants are concurrent at a point $z\in S$;*
(xxx) *each non-tangent line through $z$ is a $n$-secant.*

We shall say that $S$ is *maximal (with respect to $m$)* if no $[(n-1)(m+d)+1]$-set of type $[0,1,m+d,n]$ satisfying (x), (xx), and (xxx) contains properly $S$.

**Proposition 7.** *If $S$ is a point-set of size $(n-1)m+1$ of type $[0,1,m,n]$ in $PG(2,q)$, $q=p^r$ and $p$ prime, satisfying* (x), (xx) *and* (xxx), *then $n=p^h+1$. If, furthermore, $S$ is maximal with respect to $m$, then $m=p^d+1$, and $d\,|\,gcm[r,h]$.*

**Proof.** Let $L_1$ and $L_2$ be any two $n$-secants of $S$, and pick the $n$ points of $S\cap L_1$ together with the $q+1-n$ points of $L_2$ not lying on $S$. We obtain a set $\mathcal{B}$ of size $q+1$ contained in $L_1\cup L_2$ such that each of the remaining $(m-1)(n-1)$ points of $S$ is a nucleus of $\mathcal{B}$. Therefore, at least $n-1$ nuclei lie on every $n$-secant $\ell(\neq L_1,L_2)$ of $S$ through $z=L_1\cap L_2$. Since $\mathcal{B}_1=\mathcal{B}\cap L_1$ has size $n$, form proposition 5 we infer that $\ell$ contains no other nuclei, and hence $n-1=p^h$ follows. Furthermore $\mathbf{B}^1=S\cap L_1$, and so $\mathcal{N}(\mathcal{B})$ together with $\mathcal{B}$ forms a $p^h t$-set of type $[0,1,t,p^h]$ satisfying all three of (x), (xx), and (xxx). If $S$ is maximal, then $\mathcal{N}(\mathcal{B}) \cup \mathcal{B} = S$ and hence $m = t$. As we have seen, $t = p^d - 1$ where $d \mid gcd[r,h]$, the second part of our proposition follows.

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